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Synthesis of Multiple-Feedback Active Filters

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A synthesis technique is developed for the active RC realization of transfer functions that have all their transmission zeros on the imaginary ($j\omega$) axis. The method leads to a realization using biquadratic blocks in a multiple-feedback arrangement that is the generalization of the structure obtained from the passive, double-terminated reactive equivalents. The realization combines the easy independent tuning properties of the cascade, with the low-sensitivity characteristics of passive ladders.

I. INTRODUCTION

In tbe last two decades, the question of how to realize a prescribed rational transfer function

$$T(s) = \frac{N(s)}{D(s)} = \frac{V_{\text{out}}}{V_{\text{in}}} \tag{1}$$

by RC active structures was the subject of more than a thousand learned papers. The consensus at the present time seems to be as follows:

(*i*) The active element to be used is an operational amplifier.
(*ii*) Subnetworks (building blocks) realizing biquadratic transfer functions

$$T_i(s) = \frac{n_0^i + n_1^i s + n_2^i s^2}{d_0^i + d_1^i s + d_2^i s^2} = \frac{N_i(s)}{D_i(s)} \tag{2}$$

are constructed as intermediate steps.
(*iii*) Finally, the overall transfer function is realized as a cascade connection of tbese leading to:

$$T(s) = \prod_{i=1}^{m} T_i(s).$$

Our objective in this paper is to challenge step (*iii*) above and introduce an alternative synthesis method valid for a large class of transfer functions. This is done by adding one or more feedback loops to the cascade configuration. The effect of this will be that, while the transmission zeros of the overall system will remain the concatenation of those of the individual biquadratic blocks, this will not be true for the poles any more.

We will assume (*i*) and (*ii*) to be correct without, however, worrying about the details of the specific configuration to be used in step (*ii*) above; that is to say, our most elementary building blocks will be "black boxes" realizing transfer functions of the form given by (2). Next we select a particular structure for our investigation and motivate this selection on the basis of prior work. This is followed by the development of a synthesis method for the selected structure that simultaneously proves the generality of it.

Finally we will illustrate the method and demonstrate its advantages by an example.

To circumscribe the class of problems that we will consider, note that $T(s)$ in eq. (1) is a real, rational function of the complex frequency variable s, and therefore both $D(s)$ and $N(s)$ are real polynomials with the further restrictions:

(*i*) $D(s)$ is a strict Hurwitz polynomial, i.e., its zeros are in the open left half of the s plane,

(*ii*) the degree of $N(s)$ is not greater than that of $D(s)$, in notation:

$$\delta N \leqq \delta D.$$

We will need the additional restriction:

(*iii*) all the zeros of $N(s)$ are on the imaginary ($j\omega$) axis of the s-plane. As a consequence, $N(s)$ is either pure even or pure odd.

This restriction, while quite serious, still leaves a very large group of useful functions to be considered, especially if the reader notes that the same restriction applies to functions realized by passive lossless ladders devoid of (magnetic) coupling.

The reason we are searching for new configurations can be traced back to a paper by Orchard,[1] where he has shown by a very simple physical argument that double-terminated lossless passive structures have very low sensitivity to changes in component values inside the passband where the loss is near zero. Furthermore, it is also known that ladders are good for maintaining high out-of-band suppression. While Orchard's argument is not completely valid for voltage transfer

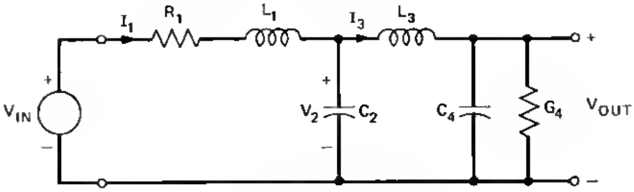

Fig. 1—Passive ladder.

functions, and therefore cannot be transferred to active realizations, improved performance *has* been obtained by simulating passive ladders by active structures.

The first results of this kind were those of Girling and Good[2] and Adams.[3] Girling and Good recognized that, say, a ladder low-pass filter of the form shown in Fig. 1 is described by the equations (also see Ref. 4):

$$I_1 = \frac{1}{R_1 + L_1 s}(V_{\text{in}} - V_2),$$

$$V_2 = \frac{1}{C_2 s}(I_1 - I_3), \tag{3}$$

$$I_3 = \frac{1}{L_3 s}(V_2 - V_{\text{out}}),$$

and

$$V_{\text{out}} = \frac{1}{C_4 s + G_4} I_3,$$

and, as such, are also the describing equations for the active RC-structure of Fig. 2. When it came to more complex filters, the structures advocated by Girling and Good become considerably more complex with nonconstant feedback and other undesirable features.

Adams,[3] working independently, went one step further and realized that the standard low-pass-to-bandpass transformation applied to both Figs. 1 and 2 would result in the very convenient realization

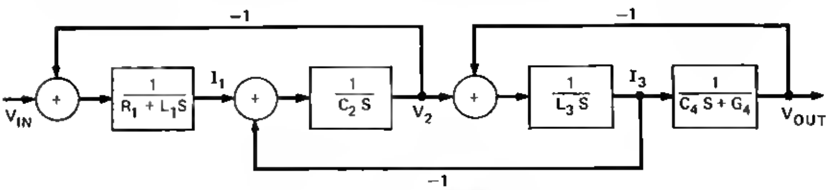

Fig. 2—Simulated passive ladder.

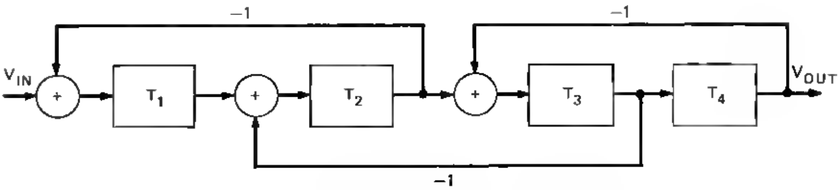

Fig. 3—Multiple feedback bandpass.

of Fig. 3 for bandpass filters. Note that all hlocks have biquadratic transfer functions T_i, and further note that T_1 and T_m (the first and last ones) have poles of finite Q, but all intermediate blocks have poles on the imaginary axis of the s-plane. These hlocks are therefore, strictly speaking, unstahle standing alone, hut the overall structure is still stahle. This hecomes less of a surprise if we note that *ideal* reactive elements (L's and C's) used in passive synthesis are also, strictly speaking, unstahle. The only difference is that natural dissipation will always push passive elements toward the stahle side of the s-plane while marginally stable active building blocks may hover over either side of the $j\omega$ axis.

Adams' numerical results were sufficiently encouraging, showing suhstantial reduction in sensitivities to start him and others on the road searching for similar realizations for more general filter functions.

There are many ways to handle this prohlem, hrute-force numerical matching, flowgraph manipulation, and matrix operations on the state-variahle equations heing some of them.[5] All of these were used hy researchers at one time or another, with mixed results. They were unahle to explain the origin and uses of multiple solutions; or if they were ahle to, they led to nonminimal realizations or realizations with much more complex feedhack and feedforward paths.[6]

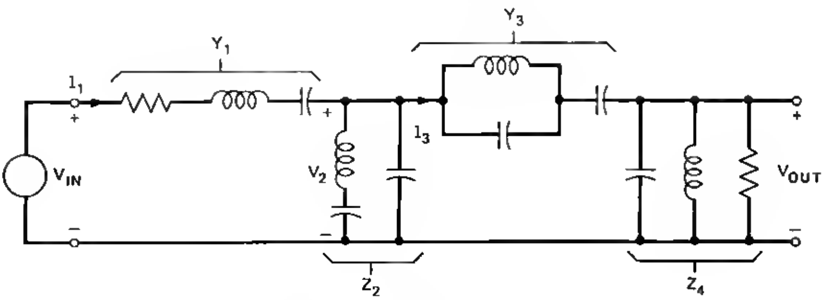

Fig. 4—Passive bandpass filter.

This author also joined in the search and, to his delight, found a group of bandpass functions where the approach of Figs. 1 and 2 was still working. Consider the passive bandpass filter shown in Fig. 4, which is a so-called minimal-inductance realization of an even degree, antimetrical filter. This network is described by the equations:

$$\begin{aligned} I_1 &= Y_1(V_{\text{in}} - V_2) \\ V_2 &= Z_2(I_1 - I_3) \\ I_3 &= Y_3(V_2 - V_{\text{out}}) \\ V_{\text{out}} &= Z_4 I_3 \end{aligned} \tag{4}$$

or, with a slight modification:

$$\begin{aligned} \frac{I_1}{s} &= \frac{Y_1}{s}(V_{\text{in}} - V_2) \\ V_2 &= sZ_2\left(\frac{I_1}{s} - \frac{I_3}{s}\right) \\ \frac{I_3}{s} &= \frac{Y_3}{s}(V_2 - V_{\text{out}}) \\ V_{\text{out}} &= sZ_4\frac{I_3}{s} \end{aligned} \tag{4a}$$

and this is identical to the equations describing the structure of Fig. 3, if we identify:

$$\begin{aligned} T_1 &= \frac{Y_1}{s}; \qquad T_3 = \frac{Y_3}{s}; \\ T_2 &= sZ_2; \qquad T_4 = sZ_4. \end{aligned} \tag{5}$$

Note that all T_i are again biquadratic, T_1 and T_4 have poles inside the left-half s-plane, while the poles of T_2 and T_3 are on the imaginary axis.

This led us to expect the general structure of the form of Fig. 3 to be in some ways canonical. Note that the restriction of all feedback coefficients to -1 does not restrict generality. For instance, the center loop feedback coefficient may be changed to $-k$ if we simultaneously multiply T_1 by k and divide T_2 by k, with no change in the overall transfer function. This procedure, in fact, can be used to adjust voltage levels inside the filter to optimize dynamic range at a later stage in the design process.

II. SYNTHESIS PROCEDURE

Consider now the general structure shown in Fig. 5 with the following assumptions:

(*i*) All T_i transfer functions are real, hiquadratic functions of s. If the overall degree is odd, T_1 or T_m will become bilinear.

(*ii*) All internal block transfer functions

$$T_2,\ T_3,\ \cdots,\ T_{m-1}$$

are even functions of s, i.e., all their singularities are located symmetrically either on the imaginary or the real axes.

(*iii*) The poles of T_1 and T_m are inside the left-half s-plane, and their numerators are either even or odd.

The synthesis prohlem can now be formulated as follows. Given a suitahly restricted overall rational transfer function of the form of eq. (1), find the biquadratic transfer functions

$$T_i(s) = \frac{N_i(s)}{D_i(s)} \qquad i = 1, 2, \cdots, m \tag{6}$$

satisfying (*i*) through (*iii*) above, such that these in the structure of Fig. 5 realize $T(s)$.

The outline of the proposed solution of this problem is as follows. First we recognize that our structure is equivalent to a (reciprocal) ladder network, if we restrict all the feedback coefficients to be -1. This restriction will he removed later in Appendix B. The synthesis problem can therefore be solved if we somehow derive a complete (impedance, admittance, or any other) set of parameters for this ladder. Once a set of parameters is obtained, the actual synthesis can follow along lines very similar to the well-known passive filter synthesis method.

Let us use our ladder ↔ multiple-feedback analogy backwards now and construct the ladder network of Fig. 6a (assuming that m is even),

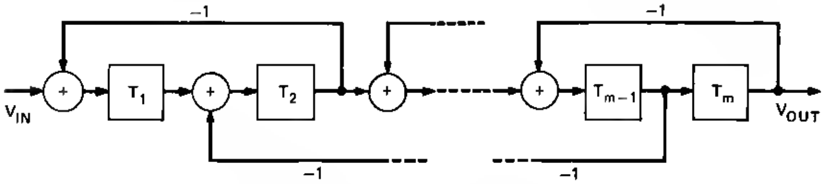

Fig. 5—General multiple-feedback structure.

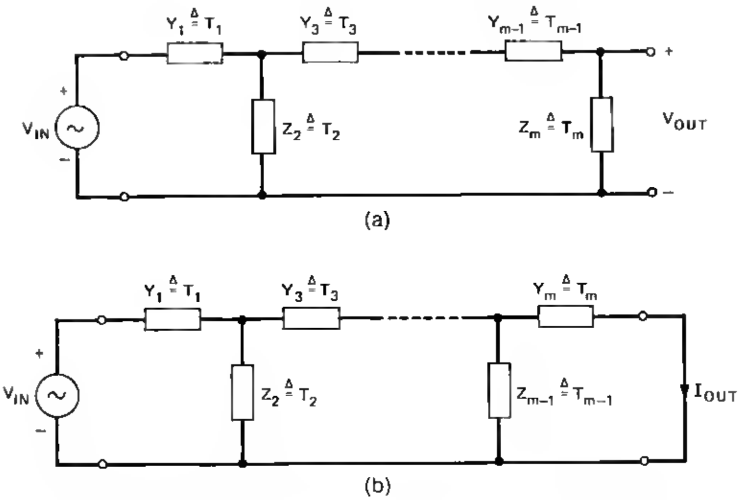

Fig. 6—(a) Equivalent pseudo-ladder for m = even. (b) Equivalent pseudo-ladder for m = odd.

where the hranches need not he realizahle passive immittances. However, the voltage transfer functions of the two structures are clearly identical and must be equal to $T = N(s)/D(s)$. Note first that obviously:

$$N(s) = \prod_{i=1}^{m} N_i(s). \tag{7}$$

The Z and Y matrices of our pseudo-ladder must he of the forms:

$$Z = \frac{1}{N_1 P}\begin{bmatrix} D & N \\ N & N_1 N_m R \end{bmatrix} \tag{8a}$$

$$Y = \frac{1}{N_m Q}\begin{bmatrix} N_1 N_m R & -N \\ -N & D \end{bmatrix} \tag{8h}$$

where P, Q and R are three unknown polynomials. These follow from Fig. 6a and the assumptions (*i*) through (*iii*) ahove. Considering degrees and parities of the polynomials in question, we see that, assuming N_1 and N_m to hoth he even:

$$\begin{aligned} &\delta D = 2m && \delta P = 2(m-1) \\ &\delta N = 2m \text{ and pure even} && \delta Q = 2(m-1) \\ &\delta R = 2(m-2) \text{ and pure even} && \end{aligned} \tag{9}$$

where δX, as before, denotes the degree of the polynomial X. That $R(s)$ must be even follows from the fact that, with a short-circuited output

$$\left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{y_{21}}{y_{11}} = -\frac{N}{N_1N_mR} = -\frac{\hat{N}}{R}$$

and since this parameter is independent of both Y_1 and Z_m, it must be pure even. Here we used the notation:

$$\hat{N} = \prod_{i=2}^{m-1} N_i. \tag{10}$$

Since $ZY = 1$, the 2×2 unit matrix, we get the determinantal relationship first:

$$DR - N_1N_m\hat{N}^2 = PQ. \tag{11}$$

Note that N_1 and N_m must be known at this stage. The significance of this is explored further in Appendix B.

In order to proceed further, we must consider the actual synthesis procedure. Consider, for instance, the parameter

$$y_{11} = \frac{N_1R}{Q}.$$

We can get Y_1 by extracting the partial fraction

$$\frac{\alpha_1 + \beta_1 s}{N_1}$$

plus a constant from $1/y_{11}$ such that the remainder has a factor N_2 in its numerator. More importantly, the remainder must be a pure even function of s, since it is now independent of both Y_1 and Z_m.

Separating $Q(s)$ into even and odd parts and performing the operations outlined, we see that $Q(s)$ must be of the form:

$$Q(s) = A(s) + \frac{k_1}{2}sR(s) \tag{12}$$

where $\delta A = 2(m-1)$ and pure even, and $k_1 = 2\beta_1$ is a constant. A similar argument applied to z_{22} shows that $P(s)$ must be of the form:

$$P(s) = B(s) + \frac{k_m}{2}sR(s) \tag{13}$$

where $\delta B = 2(m-1)$ and pure even, and k_m is a constant. Let us

now separate $D(s)$ into even $E_2(s)$ and odd $sE_1(s)$ parts:

$$D(s) = sE_1(s) + E_2(s). \tag{14}$$

Substitute (12) through (14) into the determinantal equation (11) and separate even and odd parts:

$$2E_1 \equiv k_m A + k_1 B \tag{15}$$

$$E_2 R - N_1 N_m \hat{N}^2 \equiv AB + \frac{k^2}{4} s^2 R^2 \tag{16}$$

where we introduced $k_1 k_m = k^2$.

Considering eq. (16) a quadratic in the unknown polynomial $R(s)$, this will have a polynomial solution if and only if the discriminant:

$$E_2^2 - k^2 s^2 (AB + N_1 N_m \hat{N}^2) \triangleq G^2 \tag{17}$$

is a full square for some even polynomial G. In such a case, the solution is given by:

$$R = \frac{2}{k^2 s^2}(E_2 \pm G)$$

and, in order to get a polynomial, we must select the negative sign, since from (17) we see that the constant terms of E_2 and G will be identical and hence cancel from the difference:

$$R = \frac{2(E_2 - G)}{k^2 s^2}. \tag{18}$$

Let us next express, say, B from eq. (15) and substitute it into (17):

$$E_2^2 - k^2 s^2 N_1 N_m \hat{N}^2 - 2k_m s^2 E_1 A + k_m^2 s^2 A^2 \equiv G^2. \tag{19}$$

This is again a quadratic in A, having a polynomial solution if and only if:

$$s^2 E_1^2 - E_2^2 + k^2 s^2 N_1 N_m \hat{N}^2 + G^2 = s^2 H^2 \tag{20}$$

for some even polynomial H. Note that, by eq. (17), the left side of (20) must have a double zero at $s = 0$; hence the factor s^2 on the right.

The solution to eq. (19) is then of the form:

$$A = \frac{E_1 \pm H}{k_m} \tag{21}$$

and consequently:

$$B = \frac{E_1 \mp H}{k_1}. \tag{22}$$

Now we have exchanged our three unknown polynomials R, A, and B for two new ones, G and H, both even and

$$\delta G = 2m$$
$$\delta H = 2(m - 1).$$

Let us however write eq. (20) in the form:

$$(E_2^2 - s^2E_1^2) - k^2s^2N_1N_m\hat{N}^2 \equiv (G^2 - s^2H^2). \tag{23}$$

Now introduce the notation:

$$F(s) = G(s) + sH(s)$$

and use (14) to write eq. (23) in the form:

$$\boxed{D(s)D(-s) - k^2s^2N_1N_m\hat{N}^2 \equiv F(s)F(-s)} \tag{24}$$

which is our design equation. Once we pick N_1 and N_m, the left side is known save for the constant k^2, which can be selected arbitrarily as long as the left side does not have pure imaginary roots of odd multiplicity. This condition is clearly satisfied for $k^2 = 0$ and hence, by continuity, there must be a finite range

$$0 \leqq k^2 \leqq k^2_{\max}$$

for which it is satisfied.

Our synthesis process is now straightforward and is as follows:

(*i*) Select N_1 and N_m.

(*ii*) Select k^2 arbitrarily but such that

$$D(s)D(-s) - k^2s^2N_1N_m\hat{N}^2$$

has no imaginary roots of odd multiplicity and factor this polynomial into

$$F(s)F(-s).$$

Note that $F(s)$ need not be a Hurwitz polynomial.

(*iii*) Generate G and H from:

$$F(s) = G(s) + sH(s)$$

and E_1 and E_2 from:

$$D(s) = E_2(s) + sE_1(s)$$

where E_1, E_2, G and H are all even polynomials.

(*iv*) Form A, B and R from eqs. (21), (22) and (18) respectively and P and Q from eqs. (13) and (12), giving:

$$P = \frac{(E_2 - G) + s(E_1 \mp H)}{k_1 s} = \frac{D(s) - F(\pm s)}{k_1 s} \tag{25}$$

and

$$Q = \frac{(E_2 - G) + s(E_1 \pm H)}{k_m s} = \frac{D(s) - F(\mp s)}{k_m s}. \tag{26}$$

(*v*) Select the sequence of the remaining numerators $N_i(s)$, $i = 2, 3, \cdots, m - 1$ and synthesize the network by a technique very similar to the standard "zero shifting" technique[7] used in passive reactive ladder network synthesis. The difference is merely that here we deal with even rather than odd rational fractions.

Clearly, this process has several problems.

(a) The above is only one of many cases depending on the parity of m, the parity of D, N_1, and N_m. All these cases have been analyzed, and the general design equations are summarized in Appendix A.

(b) No guarantee is available that the resulting $T_i(s)$ functions will have all positive coefficients. This is similar to the passive ladder form of realization, where all positive element values cannot, in general, be guaranteed either. However, positive coefficients for $T_i(s)$ are not necessary for realization. See Appendix D for further comments on this matter.

(c) We must make the more-or-less arbitrary choices in steps (*i*) and (*ii*) above. Our first concern is to minimize overall sensitivity. We will consider this problem in Appendix B after we have discussed the calculation of the sensitivity.

III. CALCULATION OF T_i TRANSFER FUNCTIONS

Step (*v*) above concerns the final step, that is, the calculation of the coefficients of the individual T_i transfer functions. The method is a slightly modified version of the reactive ladder synthesis, but for completeness we describe it here in some detail.

First we note that we must select an arbitrary (positive) constant k_1 and then k_m is determined from $k_1 k_m = k^2$. This constant is immaterial, since we may later multiply all odd indexed T_i by a constant and divide all even indexed T_i by the same constant. This procedure

will leave the overall transfer function unchanged if there are an even number of T_i blocks, or will multiply it by the same constant if there are an odd number of blocks in the network.

For computational simplicity, let us start by using y_{11}, which in the case of the network shown in Fig. 6a does not depend on Z_m, hence will only suffice for the calculation of $T_1, T_2, \cdots, T_{m-1}$. The last block will therefore have to be calculated from another function, say, z_{22}. From eqs. (8b), (18) and (26) we get:

$$y_{11} = \frac{N_1 R}{Q} = \frac{2N_1(E_2 - G)}{k_1 s[D(s) - F(\pm s)]}. \tag{27}$$

Inverting this, we immediately recognize that it can be written in the form:

$$(i) \quad \frac{1}{y_{11}} = \frac{k_1 s[E_2 + sE_1 - G \pm sH]}{2N_1(E_2 - G)} = \frac{k_1 s}{2N_1} + \frac{k_1}{2}\frac{s^2(E_1 \pm H)}{N_1(E_2 - G)}. \tag{28}$$

Both of these terms are known and the second is pure even; hence from this point on (that is, nearly from the beginning) we are dealing with pure even functions simplifying the work considerably. Denoting

$$(ii) \qquad z_{r1} = \frac{k_1}{2}\frac{s^2(E_1 \pm H)}{N_1(E_2 - G)} \tag{29}$$

we first note that this still has a pole pair at $s = \pm j\omega_1$, the zero of N_1, because of the factor N_1 in its denominator. We remove this by calculating:

$$(iii) \qquad \alpha_1 = N_1 z_{r1}\Big|_{s=j\omega_1} = \frac{k_1}{2}\frac{s^2(E_1 \pm H)}{E_2 - G}\Big|_{s^2=-\omega_1^2} \tag{30}$$

and calculate:

$$z_{r2} = z_{r1} - \frac{\alpha_1}{N_1} \tag{31}$$

that will not have the factor N_1 in its denominator any more. Next we select the transmission zero $\pm j\omega_2$ for the next block and calculate:

$$(iv) \qquad \beta_1 = z_{r2}\big|_{s^2=-\omega_2^2} \tag{32}$$

which leaves a remainder:

$$z_{r3} = z_{r2} - \beta_1 \tag{33}$$

that will have the factor N_2 in its numerator. At this stage we completed a full cycle and calculate the transfer function of the first block

as:

$$(v) \qquad \frac{1}{T_1} = \frac{\frac{k_1}{2}s}{N_1} + \frac{\alpha_1}{N_1} + \beta_1 = \frac{D_1}{N_1} \tag{34}$$

and bave a remainder z_{r3} that is two degrees lower. We may invert this now and go back to step (*ii*) (with an admittance function this time) and repeat this loop as many times as required. Equation (34), of course, will contain only the α_i/N_i and β_i terms giving us the required even T_i transfer functions except wben $i = 1$.

Finally, to obtain the last block as well as to check on the accuracy of computations, we repeat the same process from the output end using z_{22}. The resulting two structures sbould be identical save for a constant multiplier, that is to say, denoting the T_i transfer functions obtained from z_{22} by $\hat{T}_i$, to distinguisb them from those (T_i) obtained from y_{11}, we must have:

$$\begin{aligned} \downarrow \quad & T_1 \\ T_2 &= C\hat{T}_2 \\ T_3 &= \frac{1}{C}\hat{T}_3 \\ T_4 &= C\hat{T}_4 \\ &\vdots \\ T_{m-1} &= \frac{1}{C}\hat{T}_{m-1} \\ & C\hat{T}_m \quad \uparrow \end{aligned} \tag{35}$$

T_1 and $\hat{T}_m$ have no pairs since y_{11} and z_{22} are independent of T_m and T_1 respectively. The degree to which this set of equations is satisfied with a constant C is the numerical accuracy maintained throughout the computation. The procedure varies slightly when $N(s)$ is odd and/or the degree of $D(s)$ is odd, but only in calculating tbe first (and last) block. These variations are self-evident and hence will not be dealt with here.

Clearly, the internal sequence of zeros (those excluding N_1 and N_m) is arbitrary at this stage.

Finally, one may note that to ease the numerical accuracy problem, the commonly used transform variable[8,9] or the product method[10] may also be used.

IV. EXAMPLE

Many numerical examples have been calculated by tbe aforementioned method; here we will illustrate it witb one. The example is a handpass filter with passhand from $\omega_A = 0.8$ to $\omega_B = 1.25$ with a 0.5 dB passband ripple. The filter contains four pairs of transmission zeros at:

$$Z_{1,2} = \pm j0.25$$
$$Z_{3,4} = \pm j0.50$$
$$Z_{5,6} = \pm j2.0$$
$$Z_{7,8} = \pm j4.0.$$

The corresponding transfer function poles are at:

$$P_{1,2} = -0.028950107 \pm j0.79624226$$
$$P_{3,4} = -0.087386703 \pm j0.90192127$$
$$P_{5,6} = -0.10642659 \pm j1.0984326$$
$$P_{7,8} = -0.045602221 \pm j1.2542411.$$

Following our procedure, the resulting structure contains four biquadratic blocks with transfer functions of the form of eq. (2); the coefficients are tabulated in Tahle I.

For the synthesis, we have selected (see Appendix B):

$$N_1 = (s^2 + (0.25)^2)$$
$$N_m = N_4 = (s^2 + (4.)^2)$$

and

$$k^2 = 3.3 \times 10^{-6}$$

which is slightly below $k^2_{\max}$. Next, we selected $k_1 = k$ and $k_m = k$ and performed the synthesis in botb directions. These had a ratio

$$C = 8.00488 \times 10^6$$

that was constant to the indicated six decimal digits through all the coefficients. Rescaling the coefficients by $\sqrt{C}$ (that is to say, selecting $k_1 = k/\sqrt{C}$ and $k_m = \sqrt{C}k$) resulted in identical values up to six decimal digits in the two syntbeses, and these are the values tabulated in Table I. The computed performance of this structure is sbown in Fig. 7. All computations were done in double precision arithmetic (≈ 16 decimal digits), and the indication is tbat, for higher order cases, either the transformed variable method[11] or, preferably, the product method[10] will have to be used to avoid numerical accuracy problems.

Table I—Final Coefficients of Example

i	1	2	3	4
n_0	0.0625	4.0	0.25	16.0
n_1	0	0	0	0
n_2	1.0	1.0	1.0	1.0
d_0	7.98786	3.08873	9.31590	11.3657
d_1	2.56983	0	0	2.56983
d_2	7.74668	3.02573	9.50989	11.6790

4.1 *Practical Results*

The filter, scaled to 1-kHz center frequency, was constructed in the lahoratory, using the three-operational amplifier biquad realization[12] for the second-order huilding hlocks. The total structure (see Fig. 8) needed two additional phase inverters to get the correct signs for the feedhack loops. The component values are shown in Tahle II. The measured loss (save for a 3-dB flat difference) is also shown on Fig. 7. The agreement is the kind or hetter than the kind one usually expects in a passive realization, with no tuning.

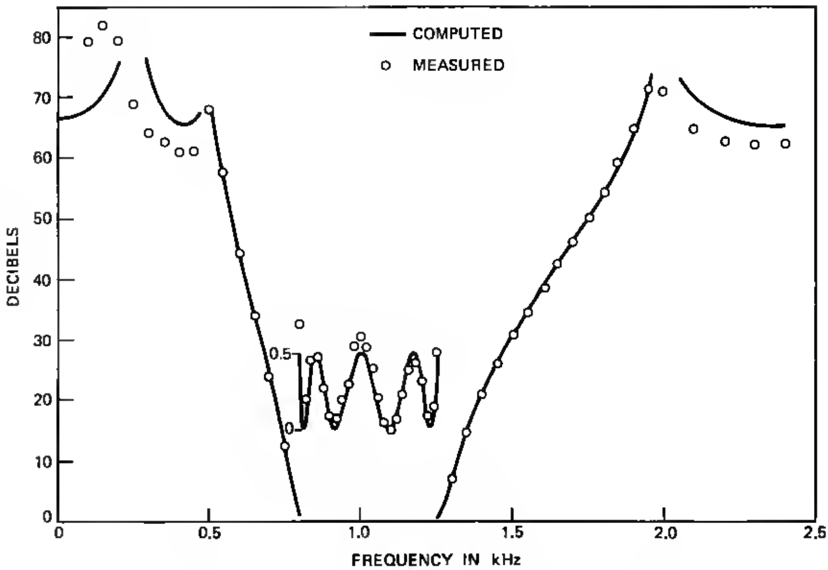

Fig. 7—Computed performance of the filter of example.

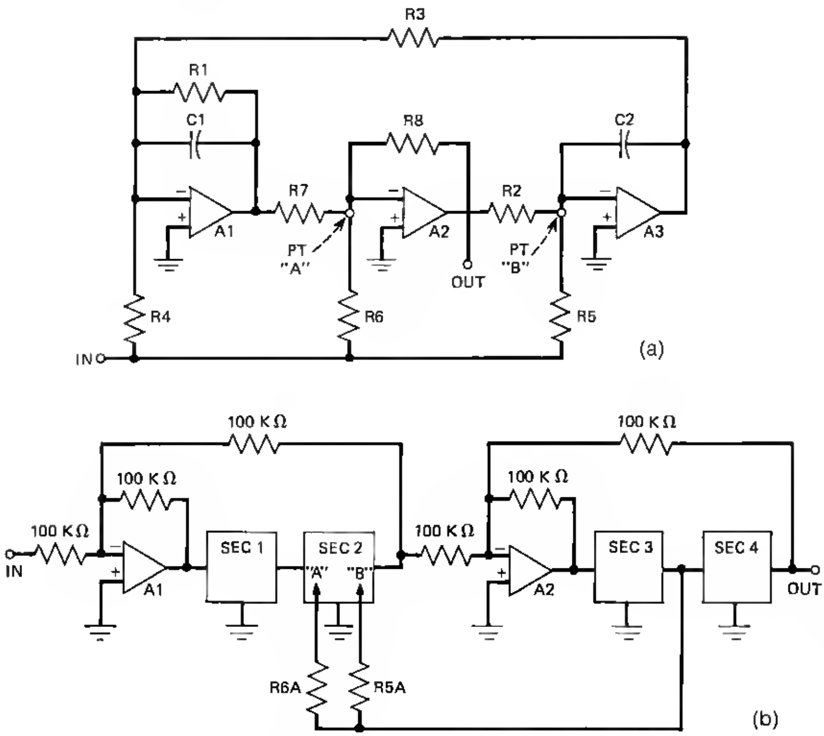

Fig. 8—(a) Basic second-order block. (b) Complete filter.

TABLE II—ELEMENT VALUES OF THE REALIZATION OF FIG. 9

	Sections 1	2	3	4
R1	48.1 kΩ	—	—	14.5 kΩ
R2	15.4 kΩ	15.8 kΩ	6.42 kΩ	3.29 kΩ
R3	16.7 kΩ	15.8 kΩ	6.42 kΩ	3.16 kΩ
R4	392. kΩ	—	—	169. kΩ
R5	1.97 MΩ	12.1 kΩ	237. kΩ	2.34 kΩ
R6	77.7 kΩ	6.04 kΩ	95.3 kΩ	17.2 kΩ
R7	9.53 kΩ	2. kΩ	10. kΩ	1.47 kΩ
R8	10. kΩ	2. kΩ	10. kΩ	1.47 kΩ
C1	0.01 μF	0.01 μF	0.025 μF	0.05 μF
C2	0.01 μF	0.01 μF	0.025 μF	0.05 μF
R5A	—	12.1 kΩ	—	—
R6A	—	6.04 kΩ	—	—

V. SENSITIVITY

There are many ways to analyze the sensitivity of different realizations, and the final component sensitivities will clearly depend on the particular realization used for the individual biquadratic blocks. In order to avoid this last step and provide some meaningful insight into the sensitivity properties of the multiple-feedback structure, we will compare it with a straightforward cascade realization.

If several biquadratic sections with transfer functions $T_i(s)$ are connected in cascade, the overall function is given by:

$$T(s) = \prod_{i=1}^{m} T_i(s) \tag{36}$$

and consequently the sensitivities are simple:

$$S_{T_i}^{T} = \frac{T_i}{T}\frac{\partial T}{\partial T_i} = 1 \quad \text{for all } i. \tag{37}$$

Clearly in our multiple-feedback structure, the situation is somewhat more complex. However, by the use of continuants,[13] we can derive closed-form expressions.

A continuant $K_n(X_1, \cdots, X_{n-2}, X_{n-1}, X_n)$ is defined by the recursion formula:

$$K_n(X_1, \cdots, X_{n-2}, X_{n-1}, X_n) = X_n K_{n-1}(X_1, \cdots, X_{n-2}, X_{n-1}) + K_{n-2}(X_1, \cdots, X_{n-2}) \tag{38}$$

and the two starting values:

$$\begin{aligned} K_0(\cdot) &= 1 \\ K_1(X_1) &= X_1. \end{aligned} \tag{39}$$

With these definitions, the overall transfer function of the multiple-feedback structure (with all feedback coefficients equal to -1) is given by:

$$\frac{1}{T} = K_m\left(\frac{1}{T_1}, \frac{1}{T_2}, \cdots, \frac{1}{T_m}\right). \tag{40}$$

Differentiating and rearranging, using some of the properties of continuants, we get the following expression for the sensitivity:

$$S_{T_i}^{T} = \frac{T_i}{T}\frac{\partial T}{\partial T_i} = \frac{T}{T_i}\frac{\partial K_m}{\partial\left(\frac{1}{T_i}\right)} = \frac{1}{1+X} \tag{41}$$

where

$$X = T_i \frac{K_{m-2}\left(\frac{1}{T_1}, \cdots, \frac{1}{T_{i-1}} + \frac{1}{T_{i+1}}, \cdots, \frac{1}{T_m}\right)}{K_{i-1}\left(\frac{1}{T_1}, \cdots, \frac{1}{T_{i-1}}\right) K_{m-i}\left(\frac{1}{T_{i+1}}, \cdots, \frac{1}{T_m}\right)}. \tag{42}$$

From this expression, we see that near a transmission zero, where $T_i \approx 0$, the sensitivity is about unity, that is, it is the same as for the cascade case. However, wherever $T_i \gg 1$, which will occur somewhere in the passband, the corresponding sensitivity becomes very small. No other feature is obvious from this equation, hence a specific case will be used to illustrate the numbers involved.

One additional problem arises in the case of the multiple-feedback structure, namely, the feedback coefficients can also vary from their nominal (-1) value. This can be taken into account by substituting a change

$$-1 \rightarrow -(1 + \epsilon)$$

in the kth feedback path by the changes:

$$T_{k+2i+1} \rightarrow (1 + \epsilon) T_{k+2i+1} \qquad i = 0, 1, \cdots$$

and

$$T_{k+2i} \rightarrow (1 + \epsilon)^{-1} T_{k+2i} \qquad i = 1, 2, \cdots.$$

This way we do not need to develop additional formulas for the sensitivities of the feedback coefficients.

The resulting sensitivity, neglecting the possible multiplier on the overall transfer function, can therefore be derived as follows: A change $\Delta F_k / F_k$ in the kth feedback coefficient has the same effect, ΔT, on the overall transfer function as the changes:

$$\frac{\Delta T_{k+i}}{T_{k+i}} = (-1)^{i+1} \frac{\Delta F_k}{F_k} \qquad i = 1, 2, \cdots, m - k. \tag{43}$$

In other words:

$$\Delta T = \sum_{i=1}^{m-k} \frac{\partial T}{\partial T_{k+i}} \Delta T_{k+i} = \sum_{i=1}^{m-k} (-1)^{i+1} \frac{\partial T}{\partial T_{k+i}} T_{k+i} \frac{\Delta F_k}{F_k} \tag{44}$$

or

$$\begin{aligned} \frac{\Delta T}{T} &= \frac{\Delta F_k}{F_k} \sum_{i=1}^{m-k} (-1)^{i+1} \frac{T_{k+i}}{T} \frac{\partial T}{\partial T_{k+i}} \\ &= \frac{\Delta F_k}{F_k} \sum_{i=1}^{m-k} (-1)^{i+1} S^T_{T_{k+i}} \end{aligned} \tag{45}$$

and finally:

$$S^T_{F_k} = \sum_{i=1}^{m-k} (-1)^{i+1} S^T_{T_{k+i}}. \tag{46}$$

Consequently, the sensitivity to a feedhack coefficient can be easily calculated in terms of the already calculated sensitivities to the second-order block transfer functions.

In order to simplify the prohlem of comparing the multiple-feedhack structure with, say, the cascade realization, we have adopted the following procedure.

We calculated the sensitivities of the overall transfer function T with respect to all the coefficients c_i of the individual hiquadratic blocks T_k. This was done for both the cascade and the leapfrog-feedback cases, and in the latter the parameters c_i also included tbe feedhack coefficients.

Next we calculated the quantity:

$$\sigma^2 = \sum_i (\mathrm{Re}\, S^T_{c_i})^2 \tag{47}$$

as a function of frequency. This quantity would give a measure of the spread of the loss if all coefficients are assumed to he statistically independent variahles with the same standard deviation.

Finally, the quantity

$$R = 10 \log_{10} \frac{\sigma_C^2}{\sigma_{MF}^2} \tag{48}$$

is calculated where the suhscript MF indicates multiple-feedhack, while the subscript C refers to the cascade configuration. R is therefore a measure of thc improvement of the sensitivity of the multiple-feedback structure over that of the cascade realization.

This quantity R is plotted as a function of frequency for the example of Section IV ahove and is shown in Fig. 9. The curve is extremely interesting, and it shows a slight deterioration in this measure of sensitivity in the stop-hand, but a spectacular improvement in the passband. The worsening in the stop-bands is minor, hut the passband improvement is suhstantially more than an order of magnitude. This behavior is very desirahle since a 1-dB spread in the stop-band is almost always immaterial, while the samc spread in the passhand loss can be disastrous.

In order to provide another comparison, a Monte Carlo yield study was made of this filter under the following assumptions.

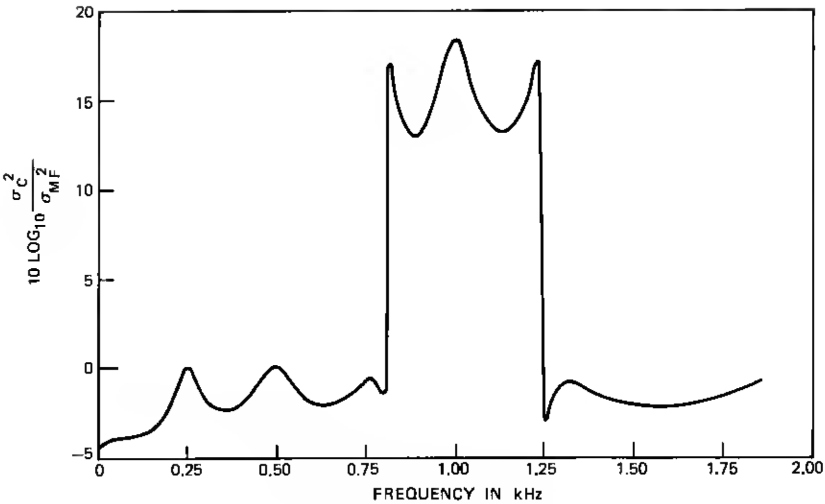

Fig. 9—Sensitivity improvement.

(*i*) Passband is acceptable if loss varies less than 1 dB from 800 Hz to 1250 Hz.
(*ii*) Stop-band is acceptable if loss is at least 60 dB from 0 to 500 Hz and from 1310 Hz up.
(*iii*) The operational amplifiers are close to ideal.
(*iv*) All passive components have the same tolerance with a flat distribution.

The resulting yields are tabulated in Table III for various tolerances. For comparison, the same calculations were performed for a cascade realization of the same filter, under the same assumptions. The results speak for themselves.

Table III—Comparison of Yields of Example and Equivalent Cascade

Tolerance	Yield %	
	MF Filter	Cascade
1.00%	44	—
0.50%	78.5	25
0.25%	100	67.5

Clearly, one cannot draw conclusions on the basis of a single example, but a general proof of the low sensitivity of the multi-feedback structure is lacking. Note also that no attempt has been made in the design of this example to minimize sensitivity apart from the crude heuristic arguments indicated in Appendix B.

VI. CONCLUSIONS

We have demonstrated that the multiple-feedback structure of Fig. 5 is a general one for transfer functions with pure imaginary zeros, and we developed a general synthesis procedure for this structure. The advantages of this realization are the low-sensitivity properties of passive ladders combined with the individually tunable transmission zeros of the cascade structure. The resulting configuration is minimal (no pole-zero cancellation occurs) and hence also stable.

The low-sensitivity properties of the realization have only been demonstrated through examples; a general proof of it is still to be found. Furthermore, apart from some heuristic arguments, no general guidelines are available for selecting the one continuously variable free parameter or to help us at the two additional places where discrete choices are to be made.

Apart from, conceivably, further optimizing the sensitivity properties of the structure, one could also use these choices, as well as additional scaling of the individual biquadratic transfer functions and the feedback coefficients, to optimize the dynamic range of the filter. All of these questions merit further investigations.

VII. ACKNOWLEDGMENTS

Many of the author's ideas were formed during lengthy discussions with several people. In particular, Paul Fleischer, Dan Hilberman, and Jimmy Tow contributed substantially to the developments reported here. Arthur Blake also bore the brunt of the programming of most of the computations. The author acknowledges his indebtedness to them and many others with great pleasure.

APPENDIX A

Summary of General Results

The results given in Section II of this paper were derived for a special case. The general results are as follows:

Given

$$T(s) = \frac{N(s)}{D(s)}$$

where the degree of D is n, and denoting:

$$m = \left[\frac{n+1}{2}\right] \text{(number of blocks)}$$

i.e., the integer part of $1/2(n+1)$ and

$$\hat{N}(s) = \sum_{i=2}^{m-1} N_i(s) \tag{49}$$

$$N(s) = N_1(s)N_m(s)\hat{N}(s) \tag{50}$$

where all $N_i(s)$ are, at most, second order, and $N_1(s)$ and $N_m(s)$ are either even or odd, while the others are strictly even. Now we define:

$$V_{1,m}(s) = \begin{cases} N_{1,m}(s) & \text{if } N_{1,m} \text{ is even} \\ -\left(1 + \dfrac{s^2}{\omega^2_{1,m}}\right) & \text{if } N_{1,m} \text{ is odd.} \end{cases} \tag{51}$$

The defining equation for

$$F(s) = G(s) + sH(s) \tag{52}$$

is:

$$\boxed{D(s)D(-s) - (-1)^m k^2 s^2 V_1(s)V_m(s)\hat{N}^2(s) \equiv F(s)F(-s)}. \tag{53}$$

The resulting pseudo-ladder is of the form shown in Fig. 6a if "m" is even, while if "m" is odd, it is as Fig. 6b, where:

$$T_i(s) = \frac{N_i(s)}{D_i(s)} \qquad i = 1, 2, \cdots, m.$$

The Z and Y matrices of these networks are of the form:

$$Z = \frac{1}{N_1P}\begin{bmatrix} D & N \\ N & N_1N_mR \end{bmatrix} \tag{54}$$

$$Y = \frac{1}{N_mQ}\begin{bmatrix} N_1N_mR & -N \\ -N & D \end{bmatrix} \tag{55}$$

for "m" even and

$$Z = \frac{1}{N_1 N_m R}\begin{bmatrix} N_m Q & N \\ N & N_1 P \end{bmatrix} \tag{56}$$

$$Y = \frac{1}{D}\begin{bmatrix} N_1 P & -N \\ -N & N_m Q \end{bmatrix} \tag{57}$$

where "m" is odd. In these equations, the still undefined quantities are the polynomial $R = R(s)$ that is always pure even, and the polynomials $P = P(s)$ and $Q(s)$ that are given by

$$\left.\begin{matrix} P(s) \\ Q(s) \end{matrix}\right\} = \frac{D(s) \pm F(\pm s)}{k_{\substack{1\\m}} s W_{\substack{1\\m}}(s)} N_{\substack{1\\m}}(s) \tag{58}$$

and

$$R(s) = \mathrm{Ev}\left\{\frac{2N_1(s)N_m(s)[D(s) \pm F(s)]}{k^2 s^2 W_1(s) W_m(s)}\right\} \tag{59}$$

where $Ev\{\quad\}$ means the even part of $\{\quad\}$,

$$W_{1,m}(s) = \begin{cases} N_{1,m}(s) & \text{if } N_{1,m}(s) \text{ is even} \\ \left(1 + \dfrac{s^2}{\omega^2_{1,m}}\right) & \text{if } N_{1,m}(s) \text{ is odd} \end{cases} \tag{60}$$

and

$$k_1 k_m = k^2.$$

Here as well as above in the definition of $V_{1,m}(s)$ the factors:

$$\left(1 + \frac{s^2}{\omega^2_{1,m}}\right) = \mathrm{Ev}\, D_{1,m}(s).$$

Hence, they are unknown initially. Note also that if n is odd, then one of the D_1 and D_m is only linear in s, i.e., the corresponding ω_1^2 or $\omega_m^2 \to \infty$.

For that case where ω_1 and/or ω_m are present, an iterative procedure must be employed, since these factors must be such that $P(s)$, $Q(s)$ and $R(s)$ are all polynomials of the correct degree.

This condition is satisfied if $(1 + s^2/\omega_1^2)$ and $(1 + s^2/\omega_m^2)$ are factors of $(E_1 + H)$ and $(E_1 - H)$ respectively, and thus can be obtained iteratively as follows. One picks the factors $(1 + s^2/\omega_1^2)$ and $(1 + s^2/\omega_m^2)$ arbitrarily, but their zeros should be in or near the passband. Then one performs the factorization of the polynomial $F(s)$ and calculates the zeros of both $(E_1 + H)$ and $(E_1 - H)$. At this stage, we can replace our arbitrary factors by the nearest factors of $(E_1 + H)$ and

$(E_1 - H)$, selecting one factor from each. A repetition of this process, or a modification of it, will converge to the desired solution.

If there is only one unknown factor involved, the iteration is somewhat simpler and any of the factors of $(E_1 \pm H)$ can he used. One must further show that, once this iteration converged, one can always select sign comhinations in the expressions for $P(s)$, $Q(s)$ and $R(s)$ such that the resulting expressions reduce to simple polynomials of the correct degrees. This can he shown directly in the limiting case $k^2 = 0$ (when, of course, no iteration is needed), and the existence of a solution for a (finite) range of nonzero $k^2 > 0$ values can then he inferred from continuity again. In order to keep the length of this paper within hounds, the details of this step are left for the reader. Finally, note that not all sign comhinations shown in eqs. (58) and (59) are allowed, hut at least one will always work.

APPENDIX B

Selection of the Parameter k^2 as Well as the End Factors N_1 and N_m

In Sections II and III we pointed out a numher of arhitrary choices the designer must make. The following comments may help in some of these decisions.

In order that eq. (53) he factorahle in the form of $F(s)F(-s)$, with $F(s)$ a real polynomial, it is necessary and sufficient that the left side have no imaginary root of odd multiplicity. Since this is clearly true for the case $k^2 = 0$ and since the roots will he continuous functions of k^2, the condition will he satisfied for a range of values

$$0 < k^2 \leqq k^2_{\max}.$$

The upper limit $k^2_{\max}$ can he calculated as follows: Since $D(s)$ cannot have a pure imaginary root, $F(s)F(-s)$ will have no imaginary root of any multiplicity as long as

$$\left.\frac{F(s)F(-s)}{D(s)D(-s)}\right|_{s=j\omega} = 1 - k^2\left[\frac{(-1)^m s^2 V_1(s)V_m(s)\hat{N}^2(s)}{D(s)D(-s)}\right]_{s=j\omega} > 0 \qquad (61)$$

or as long as

$$k^2\left[\frac{(-1)^m s^2 V_1(s)V_m(s)\hat{N}^2(s)}{D(s)D(-s)}\right]_{s=j\omega} < 1$$

and therefore this condition will he satisfied as long as

$$k^2 \leqq k_{\max}^2 = \left\{ \max_{\omega} \left[\frac{(-1)^m s^2 V_1(s) V_m(s) \hat{N}^2(s)}{D(s)D(-s)} \right]_{s=j\omega} \right\}^{-1}. \qquad (62)$$

In the limiting case $k^2 = k_{\max}^2$ there will be at least one $s = j\omega$ value where $F(s)F(-s)$ will be zero, but all of these will necessarily be of even multiplicity. Moreover, since this limiting case in passive filters corresponds to the maximum power transfer from source to load, we conjecture that it will also be optimum in our active case in the sense of providing us with the least sensitive results.

Going one step further, we note that in the passive case all the pure imaginary zeros of $F(s)$ fall inside the passband. In order to achieve this in the active case, as well as to make as many zeros of $F(s)$ approach the imaginary axis as close as possihle, the factors $V_1(s)$ and $V_m(s)$ should be selected such that

$$\left. \frac{(-1)^m s^2 V_1(s) V_m(s)}{N_1(s)N_1(-s)N_m(s)N_m(-s)} \right|_{s=j\omega} \qquad (63)$$

is as close to a *positive* constant in the passband as possible. For instance, if N_1 and N_m are both even and "m" is also even, this can be achieved by selecting $N_1(s)$ to be the lowest transmission zero below the passband and $N_m(s)$ to be the highest transmission zero above. The reason for the above requirement is simply the fact that in the passband

$$\left. \frac{N(s)N(-s)}{D(s)D(-s)} \right|_{s=j\omega} \qquad (64)$$

will usually be close to a positive constant, and therefore the product of (63) and (64) will also have this property, leading to an $F(s)F(-s)$ that is "small" inside the filter passband.

Finally, we come to the question of separating $F(s)$ out of $F(s)F(-s)$. Since $F(s)$ need not be a Hurwitz polynomial and since only a few, if any, of the zeros of $F(s)F(-s)$ will be purely imaginary (and of even multiplicity), we will have a finite numher of possihilities to choose from. At this time we have no guideline to offer, we simply note that in many cases we had to select roots with alternating real parts in order to reduce the size of the coefficients of $F(s)$, as this procedure seemed to be necessary to insure the positiveness of the final coefficient values.

Clearly, the selection of $F(s)$ will affect the final network sensitivity properties, hut further study is needed to clarify the role $F(s)$ plays in influencing the sensitivity.

APPENDIX C

Modifying the Feedback Coefficients

As mentioned in Section I of this paper, the restriction of all feedback coefficients to -1 does not restrict the generality of our structure and can be modified to scale the individual $T_k(s)$ blocks. Such a scaling is necessary to obtain the maximum overall dynamic range for the structure and can be performed simply as follows.

The T_k transfer function can be multiplied by a constant α_k if all subsequent transfer functions are modified as

$$\left.\begin{aligned} T_{k+2i} &\rightarrow \alpha_k T_{k+2i} \\ T_{k+2i-1} &\rightarrow \frac{1}{\alpha_k} T_{k+2i-1} \end{aligned}\right\} \quad i = 1, 2, \cdots .$$

Depending on the number of blocks following T_k, this will either leave the overall transfer function unchanged, or will multiply it by α_k. Also, the -1 in the $(k-1)$th feedback loop must be replaced by $-1/\alpha_k$. This procedure works for all blocks. In the case of the first block, an alternative is to multiply T_1 by a constant α_1, multiply the second feedback loop coefficient and divide T_2 by the same α_1 constant. This way one can scale each and every one of the T_k transfer functions without any effect on the overall transfer function save for a constant multiplier.

For the purpose of adjusting the dynamic range of the individual blocks, one must be able to calculate the input and output voltage levels of each block. In the case of all -1 feedback coefficients, this can be easily done by recognizing that these voltages are numerically equal to the branch "voltages" and "currents" of our pseudo-ladders of Figs. 6a or b. As such, these can be readily computed by the use of continuants again or by any other convenient way.

APPENDIX D

Comments on the Positiveness of the Coefficients

Consider now the realizability of the multiple-feedback structure. The synthesis technique given above guarantees the existence of a set of T_k biquadratic transfer functions with the specified properties for any $0 < k^2 \leqq k^2_{\max}$ and arbitrarily specified zero sequence. However, some of the coefficients may turn out to be negative. In such a case, it is preferable that one or more of the T_k blocks have an overall

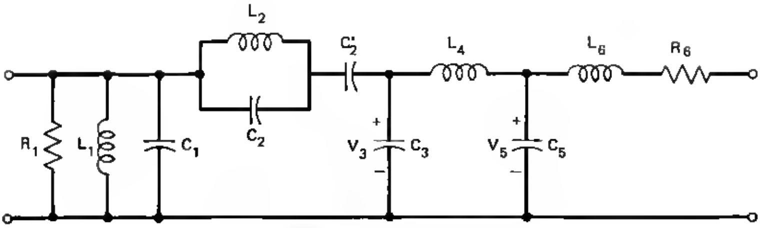

Fig. 10—Passive bandpass ladder filter.

negative sign rather than blocks having coefficients of mixed signs. We have so far been able to achieve this in all of the many examples calculated, but we had to put up with negative T_k blocks in certain cases. In particular, nonbandpass filters are likely to have negative blocks in their realization.

However, this is not a shortcoming of the method, since it appears even if the structure was derived from a passive double-terminated ladder with all positive elements. The sensitivity improvement is still realized and no instability will be generated, since our structure is minimal. Consequently, there are no pole-zero cancellations and, since the overall system poles are the zeros of $D(s)$, the system will be stable if the original requirements called for a stable transfer function.

As an example, consider the passive bandpass filter of Fig. 10. To be specific, let us select a passband from 660 Hz to 980 Hz with 0.25-dB loss ripple and transmission zeros at

$$Z_{1,2} = 0$$

$$Z_{3,4} = \infty$$

$$Z_{5,6} = \pm\, j1280 \text{ Hz}$$

$$Z_{7,8} = \infty .$$

The resulting poles normalized to the geometric center frequency are

TABLE IV—ELEMENT VALUES OF CIRCUIT IN FIG. 10

$R_1 = 1.0$ $L_1 = 0.256378$ $C_1 = 3.71840$	$C_2 = 5.46300$ $L_4 = 0.173352$
$L_2 = 0.981823$ $C_2 = 0.402048$ $C_2' = 0.556146$	$C_5 = 72.1868$ $L_6 = 0.0129235$ $R_6 = 0.00327825$

TABLE V—COEFFICIENTS OF A MULTIPLE-FEEDBACK REALIZATION OF THE PASSIVE FILTER SHOWN IN FIG. 10

	1	2	3	4
n_0	—	1.0	−1.0	0.413433
n_1	—	—	—	—
n_2	0.256378	0.394775	—	—
d_0	1.0	1.98114	5.87644	1.0
d_1	0.256378	—	—	0.236646
d_2	0.953318	1.76393	5.17359	0.932908

located at

$$P_{1,2} = -\ 0.041320302 \pm j0.81446903$$

$$P_{3,4} = -\ 0.096338325 \pm j0.94746237$$

$$P_{5,6} = -\ 0.085145570 \pm j1.1203830$$

$$P_{7,8} = -\ 0.030860810 \pm j1.2239735.$$

This filter can be realized as shown in Fig. 10 with element values (normalized to 1-ohm input impedance level and 1 rad/s center frequency) given in Table IV.

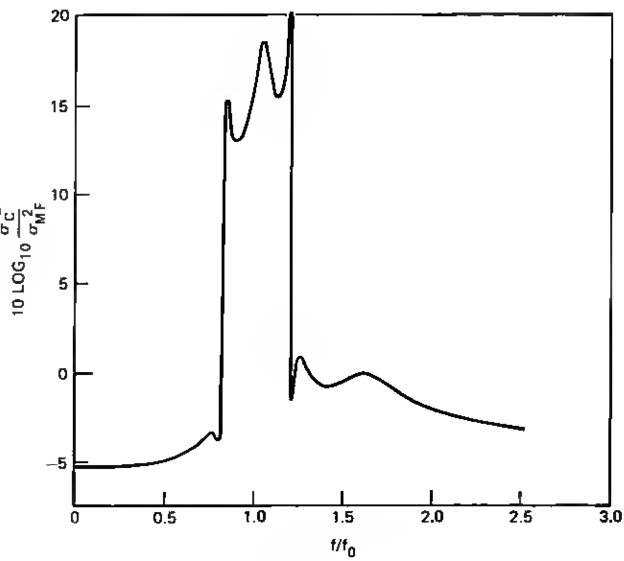

Fig. 11—Sensitivity improvement.

One can derive an active multiple-feedback equivalent of the form of Fig. 3 for this structure by writing down the branch equations as we have done in Section I, and then eliminate the variables V_3 and V_5 from them.

The resulting (normalized) coefficients are given in Table V and the most noteworthy result is that the third block has a negative sign associated with it. On the basis of Appendix C, this is equivalent to having a $+1$ (positive) feedback coefficient in the second and third loops.

Our synthesis technique was then used to generate other (equivalent) realizations, but we have failed to find one with all positive coefficients. Nevertheless, the realization shown above is quite satisfactory; the sensitivity improvement compared to the cascade realization is shown in Fig. 11.

Under what conditions are we to accept one or more negative blocks in the realization is a question that remains to be answered.

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